Introduction to Mathematics and Modeling

lecture 2

Exponentials and logarithms
This week

1. Section 1.5: exponential functions
2. Section 1.6: inverse functions and logarithms

*Model 28 circular slide rule by Concise Ltd.*

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**UNIVERSITY OF TWENTE.
Introduction to Mathematics and Modeling  Lecture 2: Exponentials and logarithms**
The inverse function: reverse engineering

- Exactly one arrow departs from every point in $D$. 

$$f: D \rightarrow Y$$

- Points in $Y$ that are not in the range of $f$ are not hit by an arrow.
- Points in the range of $f$ may be hit by more than one arrow.

Observation: If we reverse the direction of the arrows, then the result might not be a function.
The inverse function: reverse engineering

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The inverse function: reverse engineering

- Exactly one arrow departs from every point in $D$.
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- Points in the range of $f$ may be hit by more than two arrows.
The inverse function: reverse engineering

**Observation**

*If we reverse the direction of the arrows, then the result might not be a function.*
A function $f : D \to Y$ is one-to-one if $f(x_1) \neq f(x_2)$ for every $x_1$ and $x_2 \in D$ with $x_1 \neq x_2$. 

\[ f : D \to Y \]

$D$ \quad $Y$

$x_1$ \quad $f(x_1)$

$x_2$ \quad $f(x_2)$
One-to-one functions

**Definition**

A function $f : D \to Y$ is **one-to-one** if $f(x_1) \neq f(x_2)$ for every $x_1$ and $x_2 \in D$ with $x_1 \neq x_2$.

- This is equivalent with: for all $x_1$ and $x_2 \in D$ we have: if $f(x_1) = f(x_2)$ then $x_1 = x_2$. 
Definition

A function \( f : D \to Y \) is one-to-one if \( f(x_1) \neq f(x_2) \) for every \( x_1 \) and \( x_2 \in D \) with \( x_1 \neq x_2 \).

- This is equivalent with: for all \( x_1 \) and \( x_2 \in D \) we have: if \( f(x_1) = f(x_2) \) then \( x_1 = x_2 \).
- For a one-to-one function every point in \( Y \) is the end point of at most one arrow.
Example

The function \( f(x) = 2x - 1 \) is one-to-one.
Example

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Suppose \( f(x_1) = f(x_2) \), then

\[
2x_1 - 1 = 2x_2 - 1,
\]
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Example

The function \( f(x) = 2x - 1 \) is one-to-one.

- Suppose \( f(x_1) = f(x_2) \), then
  \[
  2x_1 - 1 = 2x_2 - 1, \\
  2x_1 = 2x_2, \\
  x_1 = x_2.
  \]
Example

The function \( f(x) = \frac{2x - 1}{x - 1} \) is one-to-one.
Example

The function \( f(x) = \frac{2x - 1}{x - 1} \) is one-to-one.

- Suppose \( f(x_1) = f(x_2) \), then
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  \frac{2x_1 - 1}{x_1 - 1} = \frac{2x_2 - 1}{x_2 - 1},
  \]
Example

The function \( f(x) = \frac{2x - 1}{x - 1} \) is one-to-one.

Suppose \( f(x_1) = f(x_2) \), then

\[
\frac{2x_1 - 1}{x_1 - 1} = \frac{2x_2 - 1}{x_2 - 1},
\]

\[
(2x_1 - 1)(x_2 - 1) = (2x_2 - 1)(x_1 - 1),
\]
Example

The function \( f(x) = \frac{2x - 1}{x - 1} \) is one-to-one.

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(2x_1 - 1)(x_2 - 1) = (2x_2 - 1)(x_1 - 1),
\]

\[
2x_1 x_2 - x_2 - 2x_1 + 1 = 2x_1 x_2 - x_1 - 2x_2 + 1,
\]
Example

The function \( f(x) = \frac{2x - 1}{x - 1} \) is one-to-one.

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\]

\[
2x_1 x_2 - x_2 - 2x_1 + 1 = 2x_1 x_2 - x_1 - 2x_2 + 1,
\]

\[
-x_2 - 2x_1 = -x_1 - 2x_2,
\]
Example

The function \( f(x) = \frac{2x - 1}{x - 1} \) is one-to-one.

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x_2 - 2x_1 = -x_1 - 2x_2,
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\frac{2x_1 - 1}{x_1 - 1} = \frac{2x_2 - 1}{x_2 - 1},
\]

\[
(2x_1 - 1)(x_2 - 1) = (2x_2 - 1)(x_1 - 1),
\]

\[
2x_1 x_2 - x_2 - 2x_1 + 1 = 2x_1 x_2 - x_1 - 2x_2 + 1,
\]

\[
-x_2 - 2x_1 = -x_1 - 2x_2,
\]

\[
-x_1 = -x_2,
\]

\[
x_1 = x_2.
\]
If $f$ is a one-to-one function, then a horizontal line intersects the graph of $f$ in at most one point.
Example

The function \( f(x) = 2x - 1 \) is one-to-one.
The function \( f(x) = 2x - 1 \) is one-to-one.

■ The graph of \( f \) satisfies the horizontal line test.
Example

The function \( f(x) = \frac{2x - 1}{x - 1} \) is one-to-one.
Example

The function $f(x) = \frac{2x - 1}{x - 1}$ is one-to-one.

- The graph of $f$ satisfies the horizontal line test.
Theorem

Let \( f : I \to \mathbb{R} \) be a function defined on an interval \( I \). If \( f \) is monotonous, then \( f \) is one-to-one.

\[ f(x_1) = f(x_2). \] (*

• If \( x_1 < x_2 \), then \( f(x_1) < f(x_2) \), which contradicts (*).

Conclusion: \( x_1 < x_2 \) is false.

• If \( x_1 > x_2 \), then \( f(x_1) > f(x_2) \), which contradicts (*).

Conclusion: \( x_1 > x_2 \) is false.

• There is only one option left: \( x_1 = x_2 \).
Theorem

Let $f : I \to \mathbb{R}$ be a function defined on an interval $I$. If $f$ is monotonous, then $f$ is one-to-one.

- Assume that $f$ is an increasing function.

- Assume that $f$ is a decreasing function.
Theorem

Let \( f : I \rightarrow \mathbb{R} \) be a function defined on an interval \( I \). If \( f \) is monotonous, then \( f \) is one-to-one.

- Assume that \( f \) is an increasing function.
- Let \( f(x_1) = f(x_2) \). \( (*) \)
**Theorem**

Let $f : I \rightarrow \mathbb{R}$ be a function defined on an interval $I$. If $f$ is monotonous, then $f$ is one-to-one.

- Assume that $f$ is an increasing function.
- Let $f(x_1) = f(x_2)$.  
  
  - If $x_1 < x_2$, then $f(x_1) < f(x_2)$, which contradicts ($\ast$).
    Conclusion: $x_1 < x_2$ is false.
Theorem

Let \( f : I \to \mathbb{R} \) be a function defined on an interval \( I \). If \( f \) is monotonous, then \( f \) is one-to-one.

■ Assume that \( f \) is an increasing function.

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\( (*) \)

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Monotonous functions

**Theorem**

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- Assume that \( f \) is an increasing function.
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\[
\begin{align*}
\bullet & \text{ If } x_1 < x_2, \text{ then } f(x_1) < f(x_2), \text{ which contradicts (*)}. \\
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\end{align*}
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- There is only one option left: \( x_1 = x_2 \).
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- The graph of \( f \) satisfies the horizontal line test.
Example

The function \( f(x) = 2x^2 - 1 \) is not one-to-one.

Notice that from \( f(x_1) = f(x_2) \) follows: \( x_1^2 = x_2^2 \), which does not imply \( x_1 = x_2 \).
Example

The function \( f(x) = 2x^2 - 1 \) is not one-to-one.

- Notice that from \( f(x_1) = f(x_2) \) follows: \( x_1^2 = x_2^2 \), which does not imply \( x_1 = x_2 \).

- Observe that

\[
\begin{align*}
  f(1) &= 2 \cdot 1^2 - 1 = 1, \\
  f(-1) &= 2 \cdot (-1)^2 - 1 = 1,
\end{align*}
\]

hence \( f(1) = f(-1) \).
The function \( f(x) = 2x^2 - 1 \) is not one-to-one.

- Notice that from \( f(x_1) = f(x_2) \) follows: \( x_1^2 = x_2^2 \), which does not imply \( x_1 = x_2 \).

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  hence \( f(1) = f(-1) \).

- The graph of \( f \) does not satisfy the horizontal line test.
Example

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  and
  
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  hence \( f(1) = f(-1) \).

- The graph of \( f \) does not satisfy the horizontal line test.

- One counterexample suffices.
Theorem

If $f : D \to Y$ is one-to-one, then reversing the arrows yields a function from the range of $f$ to $D$. 

$f : D \to Y$

$D$  $x$  $f(x)$  $Y$

range($f$)
The inverse function

**Theorem**

If \( f : D \to Y \) is one-to-one, then reversing the arrows yields a function from the range of \( f \) to \( D \).

\[
f : D \to Y
\]

\( x \mapsto y \)

\( D \to \text{range}(f) \)

\( Y \)

This function is called the inverse of \( f \), and is denoted as \( f^{-1} \).
The inverse function

**Theorem**

If \( f : D \to Y \) is one-to-one, then reversing the arrows yields a function from the range of \( f \) to \( D \).

\[ f^{-1} : \text{range}(f) \to D \]

- This function is called the **inverse of** \( f \), and is denoted as \( f^{-1} \).
Finding the inverse function

- If $y = f(x)$, then $x = f^{-1}(y)$.
Finding the inverse function

- If \( y = f(x) \), then \( x = f^{-1}(y) \).
- Finding the inverse means: solve the equation \( y = f(x) \) for \( x \).
Finding the inverse function

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Example

**Find the inverse of** \( f(x) = 2x - 1 \).
Finding the inverse function

- If \( y = f(x) \), then \( x = f^{-1}(y) \).
- Finding the inverse means: solve the equation \( y = f(x) \) for \( x \).

**Example**

*Find the inverse of \( f(x) = 2x - 1 \).*

- Solve \( y = 2x - 1 \) for \( x \):
  
  \[
  y = 2x - 1,
  \]

\[
\]
Finding the inverse function

- If \( y = f(x) \), then \( x = f^{-1}(y) \).
- Finding the inverse means: solve the equation \( y = f(x) \) for \( x \).

Example

**Find the inverse of** \( f(x) = 2x - 1 \).

- Solve \( y = 2x - 1 \) for \( x \):

\[
y = 2x - 1,
\]

\[
y + 1 = 2x, \quad +1
\]

We found \( f^{-1}(y) = y + \frac{1}{2} \).

Replace \( y \) by \( x \):

\( f^{-1}(x) = x + \frac{1}{2} \).
Finding the inverse function

- If \( y = f(x) \), then \( x = f^{-1}(y) \).
- Finding the inverse means: solve the equation \( y = f(x) \) for \( x \).

Example

*Find the inverse of \( f(x) = 2x - 1 \).*

- Solve \( y = 2x - 1 \) for \( x \):

\[
\begin{align*}
y &= 2x - 1, \\
y + 1 &= 2x, \\
\frac{y + 1}{2} &= x,
\end{align*}
\]

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Replace \( y \) by \( x \):

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**Example**

*Find the inverse of \( f(x) = 2x - 1 \).*

- Solve \( y = 2x - 1 \) for \( x \):
  
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  \frac{y + 1}{2} = x, \\
  x = \frac{y + 1}{2}.
  \]
Finding the inverse function

- If $y = f(x)$, then $x = f^{-1}(y)$.
- Finding the inverse means: solve the equation $y = f(x)$ for $x$.

Example

Find the inverse of $f(x) = 2x - 1$.

- Solve $y = 2x - 1$ for $x$:
  
  \[
  y = 2x - 1, \\
  y + 1 = 2x, \\
  \frac{y + 1}{2} = x, \\
  x = \frac{y + 1}{2}.
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- We found
  
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Finding the inverse function

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Example

Find the inverse of \( f(x) = 2x - 1 \).

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  y = 2x - 1,
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  y + 1 = 2x,
  \]
  \[
  \frac{y + 1}{2} = x,
  \]
  \[
  x = \frac{y + 1}{2}.
  \]
- Replace \( y \) by \( x \):
  \[
  f^{-1}(x) = \frac{x + 1}{2}.
  \]
- We found
  \[
  f^{-1}(y) = \frac{y + 1}{2}.
  \]
Example

Find the inverse of \( f(x) = \frac{2x - 1}{x - 1} \).

Solve \( x \) from the equation \( y = f(x) \):

\[ y = \frac{2x - 1}{x - 1}, \]

multiply with \( x - 1 \):

\[ y(x - 1) = 2x - 1, \]

expand:

\[ xy - y = 2x - 1, \]

collect \( x \) and \( y \):

\[ xy - 2x = y - 1, \]

factorize:

\[ x(y - 2) = y - 1, \]

divide by \( y - 2 \):

\[ x = \frac{y - 1}{y - 2}. \]

The inverse of \( f \) is \( f^{-1}(y) = \frac{y - 1}{y - 2}. \)

Replace \( y \) by \( x \):

\[ f^{-1}(x) = \frac{x - 1}{x - 2}. \]
Example

Find the inverse of \( f(x) = \frac{2x - 1}{x - 1} \).

Solve \( x \) from the equation \( y = f(x) \):

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Finding the inverse function

**Example**

Find the inverse of \( f(x) = \frac{2x - 1}{x - 1} \).

- Solve \( x \) from the equation \( y = f(x) \):

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  y = \frac{2x - 1}{x - 1},
  \]

  multiply with \( x - 1 \)

  \[
  y(x - 1) = 2x - 1,
  \]
Finding the inverse function

Example

Find the inverse of \( f(x) = \frac{2x - 1}{x - 1} \).

- Solve \( x \) from the equation \( y = f(x) \):

\[
\begin{align*}
  y &= \frac{2x - 1}{x - 1}, \\
  y(x - 1) &= 2x - 1, \\
  xy - y &= 2x - 1,
\end{align*}
\]

multiply with \( x - 1 \)
expand
Example

Find the inverse of \( f(x) = \frac{2x - 1}{x - 1} \).

- Solve \( x \) from the equation \( y = f(x) \):

\[
y = \frac{2x - 1}{x - 1},
\]

multiply with \( x - 1 \)

\[
y(x - 1) = 2x - 1,
\]

expand

\[
xy - y = 2x - 1,
\]

collect \( x \) and \( y \)

\[
xy - 2x = y - 1,
\]
Example

Find the inverse of \( f(x) = \frac{2x - 1}{x - 1} \).

Solve \( x \) from the equation \( y = f(x) \):

\[
y = \frac{2x - 1}{x - 1},
\]

\[
y(x - 1) = 2x - 1,
\]

\[
xy - y = 2x - 1,
\]

\[
x = \frac{y - 1}{y - 2},
\]

\[
x(y - 2) = y - 1,
\]

Multiplying with \( x - 1 \), expanding, collecting \( x \) and \( y \), and factorizing.
Finding the inverse function

**Example**

Find the inverse of $f(x) = \frac{2x - 1}{x - 1}$.

- Solve $x$ from the equation $y = f(x)$:

  $y = \frac{2x - 1}{x - 1}$,

  $y(x - 1) = 2x - 1$,

  $xy - y = 2x - 1$,

  $xy - 2x = y - 1$,

  $x(y - 2) = y - 1$,

  $x = \frac{y - 1}{y - 2}$. 

The inverse of $f$ is $f^{-1}(y) = \frac{y - 1}{y - 2}$.

Replace $y$ by $x$:
Example

Find the inverse of \( f(x) = \frac{2x - 1}{x - 1}. \)

- Solve \( x \) from the equation \( y = f(x) \):

  \[
  y = \frac{2x - 1}{x - 1},
  \]

  \[
  y(x - 1) = 2x - 1, \quad \text{multiply with } x - 1
  \]

  \[
  xy - y = 2x - 1, \quad \text{expand}
  \]

  \[
  xy - 2x = y - 1, \quad \text{collect } x \text{ and } y
  \]

  \[
  x(y - 2) = y - 1, \quad \text{factorize}
  \]

  \[
  x = \frac{y - 1}{y - 2}. \quad \text{divide by } y - 2
  \]

- The inverse of \( f \) is

  \[
  f^{-1}(y) = \frac{y - 1}{y - 2}.
  \]
Finding the inverse function

Example

Find the inverse of \( f(x) = \frac{2x - 1}{x - 1} \).

1. Solve \( x \) from the equation \( y = f(x) \):
   \[
y = \frac{2x - 1}{x - 1},
   \]
   
   \[
y(x - 1) = 2x - 1,
   \]
   
   \[
   xy - y = 2x - 1,
   \]
   
   \[
   xy - 2x = y - 1,
   \]
   
   \[
   x(y - 2) = y - 1,
   \]
   
   \[
   x = \frac{y - 1}{y - 2}.
   \]

2. The inverse of \( f \) is
   \[
f^{-1}(y) = \frac{y - 1}{y - 2}.
   \]

3. Replace \( y \) by \( x \):
   \[
f^{-1}(x) = \frac{x - 1}{x - 2}.
   \]
The graph of the inverse function

Let \( y = f(x) \). Then \((x, y)\) lies on the graph of \( f \).
The graph of the inverse function

- Let \( y = f(x) \). Then \((x, y)\) lies on the graph of \( f \).
- From \( y = f(x) \) follows \( x = f^{-1}(y) \), so \((y, x)\) lies on the graph of \( f^{-1} \).
Let $y = f(x)$. Then $(x, y)$ lies on the graph of $f$.

From $y = f(x)$ follows $x = f^{-1}(y)$, so $(y, x)$ lies on the graph of $f^{-1}$.

The points $(x, y)$ and $(y, x)$ are reflected across the line $y = x$. 
Let $y = f(x)$. Then $(x, y)$ lies on the graph of $f$.

From $y = f(x)$ follows $x = f^{-1}(y)$, so $(y, x)$ lies on the graph of $f^{-1}$.

The points $(x, y)$ and $(y, x)$ are reflected across the line $y = x$.

The graph of $f^{-1}$ and the graph of $f$ are symmetric with respect to the line $y = x$. 

The graph of the inverse function
If $f: D \rightarrow Y$ is not one-to-one, then discard part of $D$ such that the restriction is one-to-one.
If $f : D \to Y$ is not one-to-one, then discard part of $D$ such that the restriction is one-to-one.

The function $f : [0, \infty) \to \mathbb{R}$ defined by $f(x) = x^2$ is one-to-one.
If $f: D \to Y$ is not one-to-one, then discard part of $D$ such that the restriction is one-to-one.

The function $f: [0, \infty) \to \mathbb{R}$ defined by $f(x) = x^2$ is one-to-one.

The inverse of $f$ is the **square root**:

$$f^{-1}(x) = \sqrt{x} \quad \text{for all } x \geq 0.$$
- Draw the graphs of \( f(x) = 2x - 1 \) and its inverse \( f^{-1}(x) = \frac{x + 1}{2} \) in one picture.

**Assignment:** IMM1 - Tutorial 2.1
**Definition**

The constant function

\[ c: D \rightarrow Y \text{ is the function that assigns } c \text{ to every } x \in D. \]

\[ D = \mathbb{R}, \quad Y = \mathbb{R} \]
2.1

**Definition**

*The constant function*

$c : D \to Y$ is the function that assigns $c$ to every $x \in D$.

\[ D = \mathbb{R}, \quad Y = \mathbb{R} \]

**Definition**

*The identical map* \( \text{id} : D \to D \)

is the function that assigns $x$ to every $x \in D$.

\[ D = \mathbb{R} \]
Definition

A linear function $f : \mathbb{R} \to \mathbb{R}$ is defined as

$$f(x) = ax + b, \quad a \neq 0.$$
**Definition**

*For every integer $n$ we define*

\[ x^n = \begin{cases} 
  1 & \text{if } n = 0, \\
  x \cdot x \cdot \ldots \cdot x & \text{if is } n \geq 1, \\
  \frac{1}{x^{|n|}} & \text{if is } n < 0.
\]
Polynomials

Definition

- **Polynomials** are functions defined by

  \[ a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \]

  where \( a_0, a_1, \ldots, a_n \) are real constants with \( a_n \neq 0 \).

- The constants \( a_0, a_1, \ldots, a_n \) are called **coefficients**.

- The constant \( a_n \) is called the **leading coefficient**.

- The number \( n \) is called the **degree** of the polynomial.
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- Linear functions are polynomials of degree 1.
Definition

For every positive integer \( n \) we define the \( \sqrt[n]{x} = x^{\frac{1}{n}} \) as the inverse of \( f(x) = x^n \) where the domain of \( f \) is assumed to be

\[
[0, \infty) \quad \text{if } n \text{ is even}, \\
\mathbb{R} \quad \text{if } n \text{ is odd}.
\]
Definition

A **surd** is an *n*-th root of a rational number that cannot be simplified to a rational number.
A surd is an \( n \)-th root of a rational number that cannot be simplified to a rational number.

- These are all surds: \( \sqrt{2}, \sqrt[3]{2}, \sqrt[3]{5}, \sqrt{\frac{2}{7}}, \sqrt[3]{\frac{1}{2}} \).
- These are not surds: \( \sqrt{4}, 8^{\frac{1}{3}}, \sqrt{\frac{1}{4}} \).
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- Around 820 AD al-Khwarizmi called irrational numbers “inaudible”. This was later translated to the Latin **surdus** (“deaf” or “mute”).

(source: www.mathisfun.com)
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(source: www.mathisfun.com)

**Definition**

A **radical** is an \( n \)-th root denoted with the **radix symbol** \( \sqrt{\cdot} \).
**Surds and radicals**

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- Around 820 AD al-Khwarizmi called irrational numbers “inaudible”. This was later translated to the Latin **surdus** (“deaf” or “mute”) (source: www.mathisfun.com)

**Definition**

A **radical** is an \(n\)-th root denoted with the **radix symbol** \(\sqrt{}\).

- These are radicals: \(\sqrt{3}, \, 3^{\sqrt{8}}, \, 4^{\sqrt{\pi}}\).
- These are not radicals: \(3^{\frac{1}{2}}, \, 8^{\frac{1}{3}}, \, \pi^{1/4}\).
Definition

- *For arbitrary fractions* $\frac{p}{q}$ (with $p$ an integer and $q$ a positive integer) *we define*
  
  $$x^{\frac{p}{q}} = \left( x^{\frac{1}{q}} \right)^p.$$ 

- *If* $\alpha \in \mathbb{R}$ *is not a fraction, then* $x^\alpha$ *is defined by limits. This is beyond the scope of this course.*
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- If $\alpha \in \mathbb{R}$ is not a fraction, then $x^\alpha$ is defined by limits. This is beyond the scope of this course.

**Basic properties**

For arbitrary $x$, $y$, $\alpha$ and $\beta$ we have

1. $x^0 = 1$
2. $1^\alpha = 1$
3. $x^\alpha y^\alpha = (xy)^\alpha$
4. $x^{\alpha+\beta} = x^\alpha x^\beta$
5. $x^{\alpha-\beta} = \frac{x^\alpha}{x^\beta}$
6. $(x^\alpha)^\beta = x^{\alpha\beta}$

Some combinations of $x$, $y$, $\alpha$ and $\beta$ may cause problems!
### Definition

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$3^{1.1} \cdot 3^{0.7} = 3^{1.1+0.7} = 3^{1.8} = 3^{\frac{9}{5}} = \sqrt[5]{3^9}$
Examples

- \[ 3^{1.1} \cdot 3^{0.7} = 3^{1.1+0.7} = 3^{1.8} = 3^{\frac{9}{5}} = \sqrt[5]{3^9} \]

- \[ \frac{(\sqrt{11})^3}{\sqrt{11}} = (\sqrt{11})^{3-1} = (\sqrt{11})^2 = 11 \]
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- \( (7\sqrt{2})^{\sqrt{2}} = 7^{\frac{1}{2}} \cdot \sqrt{2} = 7^2 = 49 \)
Examples

\[ 3^{1.1} \cdot 3^{0.7} = 3^{1.1+0.7} = 3^{1.8} = 3^{\frac{9}{5}} = 5\sqrt[5]{3}^9 \]

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\[ 7^\pi \cdot 8^\pi = (7 \cdot 8)^\pi = 56^\pi \]
Examples

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- $7^\pi \cdot 8^\pi = (7 \cdot 8)^\pi = 56^\pi$

- $\left(\frac{4}{9}\right)^{\frac{1}{2}} = \frac{4^\frac{1}{2}}{9^\frac{1}{2}} = \frac{\sqrt{4}}{\sqrt{9}} = \frac{2}{3}$
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- $3^{1.1} \cdot 3^{0.7} = 3^{1.1+0.7} = 3^{1.8} = 3^{\frac{9}{5}} = \sqrt[5]{3^9}$

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- $(7\sqrt{2})^{\sqrt{2}} = 7\sqrt{2} \cdot \sqrt{2} = 7^2 = 49$

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- $\left(\frac{4}{9}\right)^{\frac{1}{2}} = \left(\frac{4^{\frac{1}{2}}}{9^{\frac{1}{2}}}\right) = \frac{\sqrt{4}}{\sqrt{9}} = \frac{2}{3}$ or $\left(\frac{4}{9}\right)^{\frac{1}{2}} = \sqrt{\frac{4}{9}} = \sqrt{\left(\frac{2}{3}\right)^2} = \frac{2}{3}$
Assignment: IMM1 - Tutorial 2.2
### Exponential behaviour: interest on a savings account

If I have 1000 Euro in a savings account and the bank gives 5% interest each year, what will be my savings after 5 years?

<table>
<thead>
<tr>
<th>Year</th>
<th>Savings (€)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1000</td>
</tr>
<tr>
<td>1</td>
<td>1000 \cdot (1.05) = 1050.00</td>
</tr>
<tr>
<td>2</td>
<td>1000 \cdot (1.05)^2 = 1102.50</td>
</tr>
<tr>
<td>3</td>
<td>1000 \cdot (1.05)^3 = 1157.63</td>
</tr>
<tr>
<td>4</td>
<td>1000 \cdot (1.05)^4 = 1215.51</td>
</tr>
<tr>
<td>5</td>
<td>1000 \cdot (1.05)^5 = 1267.28</td>
</tr>
</tbody>
</table>

![Graph showing exponential growth of savings over 5 years]
If I have 1000 Euro in a savings account and the bank gives 5% interest each year, what will be my savings after 35 years?

<table>
<thead>
<tr>
<th>Year</th>
<th>Savings (€)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1000</td>
</tr>
<tr>
<td>5</td>
<td>$1000 \cdot (1.05)^5 = 1267.28$</td>
</tr>
<tr>
<td>10</td>
<td>$1000 \cdot (1.05)^{10} = 1628.89$</td>
</tr>
<tr>
<td>15</td>
<td>$1000 \cdot (1.05)^{15} = 2078.93$</td>
</tr>
<tr>
<td>20</td>
<td>$1000 \cdot (1.05)^{20} = 2653.3$</td>
</tr>
<tr>
<td>25</td>
<td>$1000 \cdot (1.05)^{25} = 3386.35$</td>
</tr>
<tr>
<td>30</td>
<td>$1000 \cdot (1.05)^{30} = 4321.94$</td>
</tr>
<tr>
<td>35</td>
<td>$1000 \cdot (1.05)^{35} = 5516.02$</td>
</tr>
</tbody>
</table>

![Graph showing exponential growth of savings over 35 years.](image-url)
**Definition**

Let \( a > 0 \). The **exponential function** with base \( a \) is \( f(x) = a^x \).
Exponential growth and decay

**Definition**

- If a quantity $y$ depends on time and $y$ is proportional to an exponential function, then we say that $y$ grows exponentially.
- If the base is less than 1 we say that $y$ decays exponentially.

- the human population (annual growth percentage $\approx 1.14\%$),
- carbon dating (the half-life of $^{14}\text{C}$ is approximately 5730 years),
- compound interest,
- Moore’s law: the number of transistors on integrated circuits doubles approximately every two years.

**Exponential growth and decay**

If $y$ grows exponentially, then there are constants $a$ and $y_0$ such that

$$y(x) = y_0 a^x.$$
Rule of 70

Let $y$ be a quantity that grows exponentially with a growth percentage $r\%$ per time unit. If $r$ is small, the doubling time is approximately $\frac{70}{r}$ time units.
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If a capital grows with compound interest, where the interest rate is 5% per year, then the capital is doubled in $\frac{70}{5} \approx 14$ years.
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- In the example on slide 3.2 we saw that the capital, starting with 1000.– Euro, after 15 years is 2078.93 Euro.
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- If a capital grows with compound interest, where the interest rate is 5% per year, then the capital is doubled in $\frac{70}{5} \approx 14$ years.
- In the example on slide 3.2 we saw that the capital, starting with 1000.– Euro, after 15 years is 2078.93 Euro.
- The human population grows exponentially with a growth percentage of 1.14% per year, so the population doubles about every sixty years: $\frac{70}{1.14} \approx 61.4035$. 
The natural exponential function

- The derivative of an exponential function is proportional to the function itself.
The natural exponential function

- The derivative of an exponential function is proportional to the function itself.
- If \( f(x) = a^x \) then \( f'(x) = K a^x \) for some constant \( K \).

\[ e \approx 2.71828182845904523536028747135266249775724709... \]

The function \( e^x \) is called the natural exponential function.
The derivative of an exponential function is proportional to the function itself.

If \( f(x) = a^x \) then \( f'(x) = K a^x \) for some constant \( K \).

There is one specific base value for which \( K = 1 \). This base is called \( e \) and is approximately

\[
e \approx 2.71828182845904523536028747135266249775725190...
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The natural exponential function

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Let $a > 0$, then there is a constant $c \in \mathbb{R}$ such that

$$a = e^c.$$
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\[
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\]

For every \( x \) the following holds:
\[
a^x = (e^c)^x = e^{cx}
\]
Exponential growth and decay

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Exponential growth and decay

If $y$ grows exponentially, then there are constants $c$ and $y_0$ such that

$$y(x) = y_0 e^{cx}.$$
Exponential growth and decay

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If $y$ grows exponentially, then there are constants $c$ and $y_0$ such that
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- If $c > 0$, then $a > 1$ hence $y$ is exponentially growing, and $c$ is called the growth rate.
- If $c < 0$, then $a < 1$ hence $y$ is exponentially decaying, and $c$ is called the decay rate.
- The constant $y_0$ is equal to $y(0)$, and is called the initial value.
Definition

For an exponentially decaying quantity $y$ the **half-life** is defined as the time $t_h$ such that $y$ has reduced to half the original amount at $t = 0$, in other words:

$$y(t_h) = \frac{1}{2} y(0).$$
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- Let \( y(t) = y_0 e^{ct} \). From the definition follows

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y_0 e^{ct_h} = \frac{1}{2} y_0 e^{c\cdot0} = \frac{1}{2} y_0 \quad \Rightarrow \quad e^{ct_h} = \frac{1}{2} \quad \Rightarrow \quad ct_h \approx -0.6931.
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- Then for arbitrary $t$ we have

$$y(t + t_h) = y_0 e^{c(t + t_h)} = y_0 e^{ct + ct_h} = y_0 e^{ct} \cdot e^{ct_h} = \frac{1}{2} y(t).$$
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**Theorem**

An exponentially decaying quantity reduces to half the original amount over every period of time that lasts $t_h$ time units.
Exercises

- Show that an exponentially decaying quantity $y$ satisfies the following equation:

  $$y(t) = y_0 \left(\frac{1}{2}\right)^{t/t_h},$$

  where $y_0$ is the initial value and $t_h$ is the half-life.

Assignment: IMM1 - Tutorial 2.3
Definition

The logarithm with base \(a\) is the inverse of the exponential function with base \(a\):

\[ y = a^x \iff x = \log_a y \]
Logarithms are exponents

\[ R^+ = (0, \infty) \]

\[
\begin{align*}
\log_2 1 &= 0 \quad \text{because} \quad 2^0 = 1, \\
\log_2 2 &= 1 \quad \text{because} \quad 2^1 = 2, \\
\log_2 4 &= 2 \quad \text{because} \quad 2^2 = 4, \\
\log_{10} 1000 &= 3 \quad \text{because} \quad 10^3 = 1000, \\
\log_3 81 &= 4 \quad \text{because} \quad 3^4 = 81, \\
\log_9 81 &= 2 \quad \text{because} \quad 9^2 = 81, \\
\log_2 .25 &= -2 \quad \text{because} \quad 2^{-2} = \frac{1}{4} = .25.
\end{align*}
\]
The graph of $y = \log_a x$ is obtained by reflecting the graph of $y = a^x$ across the diagonal line $y = x$. 
Logarithmic laws

- \( \log_a 1 = 0 \)
Logarithmic laws

- \( \log_a 1 = 0 \)
- \( \log_a a = 1 \)
Logarithmic laws

- $\log_a 1 = 0$
- $\log_a a = 1$
- $\log_a (x y) = \log_a x + \log_a y$
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- \( \log_a a = 1 \)
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- \( \log_a \frac{x}{y} = \log_a x - \log_a y \)
- \( \log_a \frac{1}{y} = -\log_a y \)
- \( \log_a (x^p) = p \log_a x \)
Proof of the Summation Rule

**Summation rule**

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- From the definition of the logarithm with base \( a \) follows

  \[ a^u = x \quad \text{and} \quad a^v = y. \]
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- Then
  \[ a^{u+v} = a^u a^v = xy. \]
Proof of the Summation Rule

**Summation rule**

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- Define

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- From the definition of the logarithm with base \( a \) follows

  \[ a^u = x \quad \text{and} \quad a^v = y. \]

- Then

  \[ a^{u+v} = a^u a^v = xy. \]

- This implies

  \[ \log_a(x \cdot y) = u + v = \log_a x + \log_a y. \]
Transformation rule

We can write any base-$a$ logarithm in terms of a base-$b$ logarithm:

$$\log_a x = \frac{\log_b x}{\log_b a}$$
Transformation rule

We can write any base-\( a \) logarithm in terms of a base-\( b \) logarithm:

\[ \log_a x = \frac{\log_b x}{\log_b a} \]

Proof

\[ y = \log_a x \quad \implies \quad a^y = x \]
**Transformation rule**

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**Proof**

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\[
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$$\implies \quad \log_b a^y = \log_b x$$

$$\implies \quad y \log_b a = \log_b x$$

$$\implies \quad y = \frac{\log_b x}{\log_b a}$$
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$$y = \log_a x \implies a^y = x$$

$$\implies \log_b a^y = \log_b x$$

$$\implies y \log_b a = \log_b x$$

$$\implies y = \frac{\log_b x}{\log_b a}$$

Observation

Every logarithm is a scaled version of any other logarithm.
Base change for logarithms

\[
\log_2 81 = \frac{\log_3 81}{\log_3 2} = \frac{\log_3 (3^4)}{\log_3 2} = \frac{4 \log_3 3}{\log_3 2} = \frac{4}{\log_3 2}
\]
Base change for logarithms

- $\log_2 81 = \frac{\log_3 81}{\log_3 2} = \frac{\log_3 (3^4)}{\log_3 2} = \frac{4 \log_3 3}{\log_3 2} = \frac{4}{\log_3 2}$

- $\log_5 1024 = \frac{\log_2 1024}{\log_2 5} = \frac{\log_2 (2^{10})}{\log_2 5} = \frac{10}{\log_2 5}$
Base change for logarithms

- \[ \log_2 81 = \frac{\log_3 81}{\log_3 2} = \frac{\log_3 (3^4)}{\log_3 2} = \frac{4 \log_3 3}{\log_3 2} = \frac{4}{\log_3 2} \]

- \[ \log_5 1024 = \frac{\log_2 1024}{\log_2 5} = \frac{\log_2 (2^{10})}{\log_2 5} = \frac{10}{\log_2 5} \]

- \[ \log_6 3 = \frac{\log_7 3}{\log_7 6} \]
We write the logarithm with base 10 as $\log x$. 

The logarithm with base $e$ is called the natural logarithm.

Every logarithm can be rewritten as a natural logarithm:

$$\log_a x = \frac{\log x}{\log a}.$$ 

For example:

$$\log x = \ln x \ln 10 \approx 0.434 \ln x.$$
- We write the logarithm with base 10 as \( \log x \).
- We write the logarithm with base \( e = 2.71828... \) as \( \ln x \).
Logarithms with special base

- We write the logarithm with base 10 as $\log x$.
- We write the logarithm with base $e = 2.71828...$ as $\ln x$.
- The logarithm with base $e$ is called the natural logarithm.
We write the logarithm with base 10 as $\log x$.

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The logarithm with base $e$ is called the **natural logarithm**.

Every logarithm can be rewritten as a natural logarithm:

$$\log_a x = \frac{\log_e x}{\log_e a} = \frac{\ln x}{\ln a}.$$
- We write the logarithm with base 10 as \( \log x \).

- We write the logarithm with base \( e = 2.71828... \) as \( \ln x \).

- The logarithm with base \( e \) is called the **natural logarithm**.

- Every logarithm can be rewritten as a natural logarithm:

  \[
  \log_a x = \frac{\log_e x}{\log_e a} = \frac{\ln x}{\ln a}.
  \]

- For example:

  \[
  \log x = \frac{\ln x}{\ln 10} \approx 0.434 \ln x.
  \]
Definition

For an exponentially decaying quantity $y$ with decay rate $-k$ (with $k > 0$), the **half-life** is the time $t_h$ such that $y$ has reduced to half the initial amount, in other words:

$$y(t_h) = \frac{1}{2} y(0).$$
For an exponentially decaying quantity $y$ with decay rate $-k$ (with $k > 0$), the **half-life** is the time $t_h$ such that $y$ has reduced to half the initial amount, in other words:

$$y(t_h) = \frac{1}{2} y(0).$$

- Write $y(t) = y_0 e^{-kt}$. 
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- Write $y(t) = y_0 e^{-kt}$.
- From slide 34 (with $c = -k$):

  $$e^{-kt_h} = \frac{1}{2} \implies e^{kt_h} = 2 \implies kt_h = \ln 2 \approx 0.6931.$$
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- Write $y(t) = y_0 e^{-kt}$.
- From slide 34 (with $c = -k$):
  \[
e^{-kt_h} = \frac{1}{2} \implies e^{kt_h} = 2 \implies kt_h = \ln 2 \approx 0.6931.
  \]
- The relation between decay rate and half-life is described by the following equations:

\[
\begin{align*}
t_h &= \frac{\ln 2}{k} \\
k &= \frac{\ln 2}{t_h}
\end{align*}
\]
Assignment: IMM1 - Tutorial 2.4